Math 055 Exam 3
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. (5 pts) Solve the system by finding eigenvalues and eigenvectors.

$$\frac{dx}{dt} = 3x - 2y$$
$$\frac{dy}{dt} = 2x + 3y$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 13 = 0 \Rightarrow \lambda^2 - 6\lambda + 9 = -4$$

$$\lambda = 3 \pm 2i$$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \vec{K} = \vec{0} \qquad R_2 = iR_1$$

Using R_2: $k_1 = ik_2$ Let $k_2 = -i$

Then $\vec{K} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$

$$\vec{X}(t) = e^{3t}\left\{ c_1 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t \right] + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t \right] \right\}$$

$$\begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{matrix} c_1 \cos 2t + c_2 \sin 2t \\ c_1 \sin 2t - c_2 \cos 2t \end{matrix}$$

2. (4 pts) The eigenvalue for the system $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}\mathbf{X}$ is $\lambda = 2$ (multiplicity 2). Use this information to find the general solution to the system.

$\lambda = 2$ $\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\vec{K} = \vec{0} \Rightarrow k_1 = \frac{1}{2}k_2$

Let $k_2 = 2$. Then $\vec{K} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\vec{P} = \vec{K} \longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right]$

$R_2 = 2R_1$. Using R_1, $-2p_1 + p_2 = 1$.

Let $p_2 = 0$. Then $p_1 = -\frac{1}{2}$.

$\vec{X} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}te^{2t} + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}e^{2t}\right]$

3. (3 pts) The eigenvalue for the system $\mathbf{X}' = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\mathbf{X}$ is $\lambda = 2$ (mult. 3). Use this information and the information below to find the general solution to the system.

$\mathbf{K} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{X}(t) = c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}e^{2t}\right]$$
$$+ c_3\left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}e^{2t}\right]$$

OR

$$\vec{X}(t) = c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} t \\ -1 \\ t \end{pmatrix}e^{2t} + c_3\begin{pmatrix} \frac{1}{2}t^2 - 1 \\ -t \\ \frac{1}{2}t^2 \end{pmatrix}e^{2t}$$

4. (5 pts) The complementary function of a system is given. Use variation of parameters to find the particular solution of the system.

$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 2e^{4t} \\ 0 \end{pmatrix}, \quad \mathbf{X}_c(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{3t}$

$$\Phi = \begin{pmatrix} e^{t} & e^{3t} \\ e^{t} & -e^{3t} \end{pmatrix} \Rightarrow \Phi^{-1} = -\frac{1}{2e^{4t}}\begin{pmatrix} -e^{3t} & -e^{3t} \\ -e^{t} & e^{t} \end{pmatrix}$$

$$\int \Phi^{-1}\vec{F}\,dt = \frac{1}{\cancel{2}}\int \begin{pmatrix} e^{-t} & e^{-t} \\ e^{-3t} & -e^{-3t} \end{pmatrix}\begin{pmatrix} \cancel{2}e^{4t} \\ 0 \end{pmatrix}dt$$

$$= \int \begin{pmatrix} e^{3t} \\ e^{t} \end{pmatrix}dt = \begin{pmatrix} \frac{1}{3}e^{3t} \\ e^{t} \end{pmatrix}$$

$$\Phi\int \Phi^{-1}\vec{F}\,dt = \begin{pmatrix} e^{t} & e^{3t} \\ e^{t} & -e^{3t} \end{pmatrix}\begin{pmatrix} \frac{1}{3}e^{3t} \\ e^{t} \end{pmatrix}$$

$$\vec{X}_p = \begin{pmatrix} \frac{1}{3}e^{4t} + e^{4t} \\ \frac{1}{3}e^{4t} - e^{4t} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix}e^{4t}$$

5. (10 pts) Find two power series solutions of the differential equation about the ordinary point $x = 0$. Please include three terms for each solution if possible.
$y'' + xy' + 2y = 0$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

Exponents:

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} 2c_k x^k = 0$$

Indices:

$k=0$

$$2c_2 + 2c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + (k+2)c_k\right] x^k = 0$$

$$c_2 = -c_0, \quad c_{k+2} = -\frac{1}{k+1} c_k, \quad k = 1, 2, 3, \ldots$$

$k=1 \quad c_3 = -\frac{1}{2}c_1, \quad k=2 \quad c_4 = -\frac{1}{3}c_2 = \frac{1}{3}c_0$

$k=3 \quad c_5 = -\frac{1}{4}c_3 = \frac{1}{8}c_1$

$$y = c_0\left(1 - x^2 + \frac{1}{3}x^4 + \cdots\right) + c_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 + \cdots\right)$$

$y_1 = 1 - x^2 + \frac{1}{3}x^4 + \cdots, \quad y_2 = x - \frac{1}{2}x^3 + \frac{1}{8}x^5 + \cdots$

6. (6 pts) Consider the differential equation $x^2y'' + xy' - (4+x)y = 0$. Use the method of Frobenius to find the indicial roots and recurrence relation (in terms of r and k) for the equation. (That is, find the values of r and the recurrence relation and then stop. Don't plug the r-values into the recurrence relation.)

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty}4c_n x^{n+r} - \sum_{n=0}^{\infty}c_n x^{n+r+1} = 0$$

Combine the first 3 sigmas; express everything in terms of x^{k+r}: $\displaystyle\sum_{k=0}^{\infty}\left[(k+r)(k+r-1+1)-4\right]c_k x^{k+r} - \sum_{k=1}^{\infty}c_{k-1}x^{k+r} = 0$

$k=0$: $[r(r)-4]c_0 x^r = 0 \Rightarrow r = \pm 2$

Combine both sigmas: $\displaystyle\sum_{k=1}^{\infty}\left\{\left[(k+r)^2-4\right]c_k - c_{k-1}\right\}x^{k+r} = 0$

Recurrence relation: $c_k = \dfrac{1}{(k+r)^2-4}c_{k-1}$, $k=1,2,3,\ldots$

with $r = \pm 2$.